



# An analytical model for composite tubes with bend–twist coupling



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## ABSTRACT

A simple analytical model is derived for composite tubes subjected to bending and torsion. The model is applied to a novel design of composite tube that exhibits bend–twist coupling such that the shear center of the cross section is away from the tube axis. Analytical expressions are derived for the shear center distance – distance of the shear center from the tube axis. Finite element analysis of the tubes is performed using shell elements. The agreement between analytical and numerical results is excellent. It is found that the shear center distance is independent of the radius of the tube but proportional to the length of the tube and it is also a function of coefficient of mutual influence of the composite material.

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## 1. Introduction

Elastic coupling including bend–twist (BT) coupling in composite structures has been a subject of research since the advent of fiber reinforced composite materials. Earliest interest in exploiting the coupling was in aero-elastic tailoring of aircraft wings, e.g., [1,2]. Most of the early works on understanding the effects of BT coupling were concerned with laminated composites [2]. A discussion on various elastic coupling in laminated composites can be found in [3]. Recently there has been interest in implementing the BT coupling in wind-turbine blades [4] and in composite marine propellers [5]. While most research on elastic coupling has focused on flat plates or shallow shells, not much work has been done in using the coupling effects in circular tubes.

Tubular structures are efficient when the loading is a combination of bending and torsion. Tubes made of homogeneous materials, either isotropic or anisotropic, are axisymmetric and hence there will be no coupling between flexural and torsional deformations. However, there are applications wherein a transverse force has to be applied to a tubular lever eccentrically (see Fig. 1) without causing the tube to twist. That is, the shear center has to be away from the axis of the tube. There may be other requirements such as the tube should have equal bending rigidity in all planes.

Recently Rohde et al. [6] has proposed a novel composite tube design which yields prescribed bend–twist coupling. The design is described in detail in the Results and Discussions section. In this paper we present a beam theory for an anisotropic composite tube subjected to combined bending and torsion to demonstrate

bend–twist coupling in the aforementioned tube. Behavior of the tube for various types of anisotropy is discussed. The analytical results are verified using finite element analysis. A simple formula is derived for the shear center distance – distance of shear center from the axis of the tube – which can be used to optimize the design. Some of the examples considered are of tubes with non-zero shear center distance, i.e., the shear center does not coincide with the center of the tube.

It is worth pointing out that Rao and Chan [7] developed an analytical model for the analysis of laminated tubes subjected to axial force and twisting moment. They modified the lamination theory to account for the ply stiffness of a differential element along the circumference of the tube using appropriate transformation. Then the stiffness of the tube was obtained by integrating the stiffness around the circumference. In the present approach suitable assumptions are made about the deformation of the tube and the shear stress distribution that leads to an independent beam theory for composite tubes.

## 2. Analytical model

### 2.1. A beam theory for combined bending and torsion of tubes

Consider a thin-walled tube with the tube axis parallel to the  $x$ -axis. The mean radius of the tube is  $R$  and the wall thickness  $h \ll R$ . The tube is made of two anisotropic materials – top half ( $0 < \theta < \pi$ ) is made of Material 1 and the bottom half ( $\pi < \theta < 2\pi$ ) is of Material 2. We assume that the tube is in a state of plane stress normal to the radial direction  $n$  (see Fig. 2) such that  $\sigma_{nn} = \tau_{nx} = \tau_{ns} = 0$ . Furthermore, we assume the hoop or circumferential stress  $\sigma_{ss} = 0$ . Thus the two significant stresses are the

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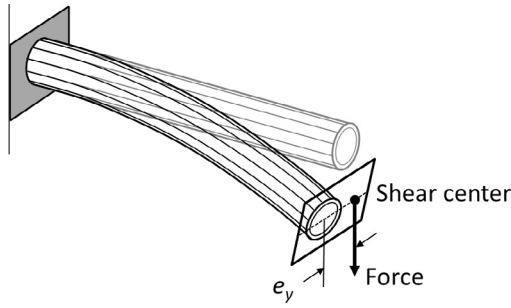


Fig. 1. A composite tube with shear center away from the tube axis. The tube is subjected to an eccentric transverse force without twisting.

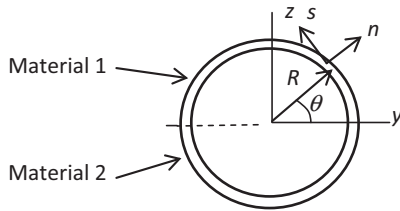


Fig. 2. Cross section of the tube and the coordinate system.

axial stress  $\sigma_{xx}$  and the shear stress  $\tau_{xs}$ . We assume that the tube deforms such that plane sections remain plane and normal to the tube axis as in Bernoulli–Euler beam theory. As will be seen later this assumption works well for the thin-walled, long tubes considered in this study. Then the displacement field can be written as

$$\begin{aligned} u(x, y, z) &= u_0(x) - y \frac{dv_0}{dx} - z \frac{dw_0}{dx} \\ v(x, y, z) &= v_0(x) \\ w(x, y, z) &= w_0(x) \end{aligned} \tag{1}$$

where  $u_0$ ,  $v_0$  and  $w_0$  are the deflections of the beam axis. The axial strain takes the form

$$\epsilon_{xx} = \frac{\partial u}{\partial x} = \frac{\partial u_0}{\partial x} - y \frac{d^2 v_0}{dx^2} - z \frac{d^2 w_0}{dx^2} = \epsilon_{x0} + y\kappa_y + z\kappa_z \tag{2}$$

where  $\kappa_y$  and  $\kappa_z$  are the curvatures. We do not make any specific assumption about the rotation of the cross section except the average rotation of the cross-section about the  $x$ -axis is denoted by  $\psi_x$ . But we assume the shear stress is uniform and given by

$$\tau_{xs} = \frac{T}{2\pi R^2 h} = \tau_0 \tag{3}$$

where  $T$  is the torque acting on a cross section.

Though the above assumption about the shear stress distribution seems to be trivial, it is important in the present context. In traditional torsion theories the plane cross section of the tube is assumed to rotate as a rigid body leading to constant  $\gamma_{xs}$  shear strain along the circumference of the tube. When the material is also homogenous it leads to constant shear stress  $\tau_{xs}$  which is consistent with the assumption in Eq. (3) above. However, when the tube is made of two different materials, the shear stress  $\tau_{xs}$  has to be continuous at the interface between the two materials, but the shear strain  $\gamma_{xs}$  does not have to be continuous at the interface. That is why no specific kinematic assumption is made regarding the rotation of the tube.

The constitutive relation for both materials can be written in the form [8]

$$\begin{Bmatrix} \epsilon_{xx} \\ \gamma_{xs} \end{Bmatrix} = \begin{bmatrix} \frac{1}{E_x} & \frac{\eta_{xs,x}}{G_{xs}} \\ \frac{\eta_{x,xs}}{E_x} & \frac{1}{G_{xs}} \end{bmatrix} \begin{Bmatrix} \sigma_{xx} \\ \tau_{xs} \end{Bmatrix} = \begin{bmatrix} \bar{S}_{11} & \bar{S}_{16} \\ \bar{S}_{16} & \bar{S}_{66} \end{bmatrix} \begin{Bmatrix} \sigma_{xx} \\ \tau_{xs} \end{Bmatrix} \tag{4}$$

where  $E$  and  $G$  are Young's modulus and shear modulus, respectively,  $\eta$  is the coupling term called coefficient of mutual influence of the material, and  $\bar{S}_{ij}$  are the transformed compliance terms of the lamina. Then from (2)–(4) we obtain

$$\sigma_{xx} = E_x \epsilon_{xx} - \eta_{x,xs} \tau_{xs} = E_x (\epsilon_{x0} + y\kappa_y + z\kappa_z) - \eta_{x,xs} \tau_0 \tag{5}$$

The force and bending moment resultants are defined as

$$\begin{aligned} (P, M_y, M_z) &= \int_A \sigma_{xx}(1, z, -y) dA \\ &= \int_0^{2\pi} \sigma_{xx}(1, R \sin \theta, -R \cos \theta) R h d\theta \end{aligned} \tag{6}$$

where the integration is performed over the cross section of the tube. Substituting for  $\sigma_{xx}$  from (5) we obtain

$$\begin{aligned} (P, M_y, M_z) &= \int_0^\pi [E_x^{(1)} (\epsilon_{x0} + R \cos \theta \kappa_y + R \sin \theta \kappa_z) - \eta_{x,xs}^{(1)} \tau_0] \\ &\quad \times (1, R \sin \theta, -R \cos \theta) R h d\theta + \int_\pi^{2\pi} [E_x^{(2)} (\epsilon_{x0} + R \cos \theta \kappa_y \\ &\quad + R \sin \theta \kappa_z) - \eta_{x,xs}^{(2)} \tau_0] (1, R \sin \theta, -R \cos \theta) R h d\theta \end{aligned} \tag{7}$$

where the superscripts (1) and (2) refer to the two materials. Performing the integration we obtain relations between the force and moment resultants and deformations:

$$\begin{bmatrix} \bar{E}_x A & \frac{AR}{\pi} \Delta E_x \\ \frac{AR}{\pi} \Delta E_x & \bar{E}_x I \end{bmatrix} \begin{Bmatrix} \epsilon_{x0} \\ \kappa_z \end{Bmatrix} = \begin{Bmatrix} P \\ M_y \end{Bmatrix} + \begin{Bmatrix} A \bar{\eta}_{x,xs} \\ 2R^2 h \Delta \eta_{x,xs} \end{Bmatrix} \tau_0 \tag{8}$$

$$\bar{E}_x I \kappa_y = -M_z$$

In the above equations  $\bar{E}_x$  is the average Young's modulus given by  $\bar{E}_x = (E_x^{(1)} + E_x^{(2)})/2$ ,  $\Delta E_x$  is the difference in the Young's moduli,  $\Delta E_x = (E_x^{(1)} - E_x^{(2)})$ . Similarly  $\bar{\eta}_{x,xs} = (\eta_{x,xs}^{(1)} + \eta_{x,xs}^{(2)})/2$  and  $\Delta \eta_{x,xs} = (\eta_{x,xs}^{(1)} - \eta_{x,xs}^{(2)})$ ,  $A = 2\pi R h$  is the cross sectional area and  $I$  is the second moment of inertia given by  $I = \pi R^3 h$ . Eq. (8) can be inverted to obtain

$$\begin{Bmatrix} \epsilon_{x0} \\ \kappa_z \end{Bmatrix} = \frac{1}{K} \begin{bmatrix} \bar{E}_x I & \frac{-AR}{\pi} \Delta E_x \\ \frac{-AR}{\pi} \Delta E_x & \bar{E}_x A \end{bmatrix} \left( \begin{Bmatrix} P \\ M_y \end{Bmatrix} + \begin{Bmatrix} \bar{\eta}_{x,xs}/R \\ \Delta \eta_{x,xs}/\pi \end{Bmatrix} T \right)$$

$$\kappa_y = -M_z / \bar{E}_x I \tag{9}$$

where  $K = \bar{E}_x^2 A I - \left( \frac{AR \Delta E_x}{\pi} \right)^2$

Thus one can calculate the deformations from the force and moment resultants. The deflections can be obtained by integrating the strains and curvatures as will be shown in the examples. From Eq. (9) it is evident that a torque  $T$  can result in curvature  $\kappa_z$  causing deflection of the tube in the  $z$ -direction. It is interesting to note that the torque  $T$  does not cause curvature  $\kappa_y$  and this is due to symmetry of the cross section about the  $z$ -axis.

### 2.2. Angle of twist

The average torsional rotation  $\psi_x$  is calculated as follows. Let us define the average unit angle of twist  $\bar{\phi} = d\psi_x/dx$ . The shear strain can be written as

$$\gamma_{xs} = \frac{\partial u_s}{\partial x} + \frac{1}{R} \frac{\partial u}{\partial \theta} = R \bar{\phi} + \frac{1}{R} \frac{\partial u}{\partial \theta} \tag{10}$$

where  $u_s$  is the displacement at a point in the tangential direction. The first term on the RHS of the above equation has been written as  $\partial u_s / \partial x = R\phi$ , where  $\phi(s)$  is the unit angle of twist at a given location  $s$ . Then the average unit angle of twist is obtained as

$$\begin{aligned} \bar{\phi} &= \frac{1}{2\pi} \int_0^{2\pi} \phi d\theta = \frac{1}{2\pi} \int_0^{2\pi} \frac{\gamma_{xs}}{R} d\theta - \frac{1}{2\pi} \int_0^{2\pi} \frac{1}{R^2} \frac{\partial u}{\partial \theta} d\theta \\ &= \frac{1}{2\pi R} \int_0^{2\pi} \gamma_{xs} d\theta \end{aligned} \quad (11)$$

Note the second integral in the above equation vanishes as it is the contour integration of an exact differential. From the constitutive relation (4) the shear strain at a point on the circumference of the tube can be written in terms of stresses as

$$\gamma_{xs} = \frac{\eta_{x,xs}}{E_x} \sigma_{xx} + \frac{\tau_{xs}}{G_{xs}} \quad (12)$$

From (12) it is clear that the shear strain  $\gamma_{xs}$  will not be continuous across the interface between Materials 1 and 2 because of the difference in elastic constants and also due to the difference in bending stress  $\sigma_{xx}$ . That is why we calculate the average rotation by integrating the unit angle of twist along the circumference of the tube.

Substituting for  $\sigma_{xx}$  from (5) into (12) and then substituting for  $\gamma_{xs}$  from (12) into (11) the average unit angle of twist can be derived as

$$\bar{\phi} = \frac{\bar{\eta}_{x,xs}}{R} \varepsilon_{x0} + \frac{\Delta \eta_{x,xs}}{\pi} \kappa_z + \frac{1}{R} \left[ -\left( \frac{\eta_{x,xs}^2}{E_x} \right) + \frac{1}{G_{xs}} \right] \tau_0 \quad (13)$$

where

$$\left( \frac{\eta_{x,xs}^2}{E_x} \right) = \frac{1}{2} \left( \frac{(\eta_{x,xs}^{(1)})^2}{E_x^{(1)}} + \frac{(\eta_{x,xs}^{(2)})^2}{E_x^{(2)}} \right) \frac{1}{G_{xs}} = \frac{1}{2} \left( \frac{1}{G_{xs}^{(1)}} + \frac{1}{G_{xs}^{(2)}} \right) \quad (14)$$

Note that  $\bar{G}_{xs}$  is the harmonic mean of the shear moduli of the two materials. Once the deformations  $\varepsilon_{x0}$  and  $\kappa_z$  are calculated from (9), the unit angle of twist can be calculated using (13). Using Eq. (9) one can express  $\varepsilon_{x0}$  and  $\kappa_z$  in Eq. (13) in terms of force and moment resultants  $P$ ,  $M_y$  and  $T$ . Thus it is obvious that a bending moment  $M_y$  about the  $y$ -axis can cause twisting in the tube demonstrating the bend–twist coupling.

### 2.3. Shear stress due to transverse forces

The present formulation is based on Euler–Bernoulli beam theory and hence the shear stresses due to transverse forces  $V_y$  and  $V_z$  are not accounted for. Only the shear stresses due to the torque are included. The transverse shear stresses can be recovered from  $\sigma_{xx}(x)$  using the equilibrium equation as in classical mechanics of materials. First consider the shear force  $V_z$ . The bending moment created will be  $M_y$  and they are related by  $dM_y/dx = V_z$ . Integrating the equilibrium equation we obtain

$$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \tau_{xs}}{\partial s} = 0 \Rightarrow \tau_{xs}(s) = \tau_{xs}(0) - R \int_0^\theta \frac{\partial \sigma_{xx}}{\partial x} d\theta \quad (15)$$

Substituting for  $\sigma_{xx}$  from (5) we obtain

$$\tau_{xs}(s) = \tau_{xs}(0) - R \int_0^\theta \frac{\partial (E_x(\varepsilon_{x0} + Z\kappa_z))}{\partial x} d\theta \quad (16)$$

Again substituting for  $\varepsilon_{x0}$  and  $\kappa_z$  from (9) in terms of  $M_y$  and using  $dM_y/dx = V_z$ :

$$\begin{aligned} \tau_{xs}(s) &= \tau_{xs}(0) - \frac{V_z R}{K} \int_0^\theta E_x \left( \frac{-AR}{\pi} \Delta E_x + \bar{E}_x Az \right) d\theta \\ &= \tau_{xs}(0) - \frac{V_z R^2 A}{K} \int_0^\theta E_x \left( \frac{-\Delta E_x}{\pi} + \bar{E}_x \sin \theta \right) d\theta \end{aligned} \quad (17)$$

Performing the above integration we obtain an expression for shear stresses due to  $V_z$ :

$$\begin{aligned} \tau_{xs}(\theta) &= \tau_{xs}(0) - \frac{V_z R^2 A E_x^{(1)}}{K} \left( \frac{-\Delta E_x}{\pi} \theta + (-\cos \theta + 1) \bar{E}_x \right), \\ &0 < \theta < \pi = \tau_{xs}(0) - \frac{2V_z R^2 A E_x^{(1)} E_x^{(2)}}{K} - \frac{V_z R^2 A E_x^{(2)}}{K} \\ &\left( \frac{-\Delta E_x}{\pi} (\theta - \pi) - (1 + \cos \theta) \bar{E}_x \right), \quad \pi < \theta < 2\pi \end{aligned} \quad (18)$$

Similarly we can derive an expression for transverse shear stress due to shear force  $V_y$  as:

$$\tau_{xs}(s) = \tau_{xs}(0) - R \int_0^\theta \frac{\partial (E_x y \kappa_y)}{\partial x} d\theta \quad (19)$$

Substituting for  $\kappa_y$  from (9) and using  $dM_z/dx = -V_y$ :

$$\begin{aligned} \tau_{xs}(\theta) &= \tau_{xs}(0) - \frac{V_y R^2}{E_x I} \int_0^\theta E_x \cos \theta d\theta \\ &= \tau_{xs}(0) - \frac{V_y R^2 E_x^{(1)} \sin \theta}{E_x I}, \quad 0 < \theta < \pi \\ &= \tau_{xs}(0) - \frac{V_y R^2 E_x^{(2)} \sin \theta}{E_x I}, \quad \pi < \theta < 2\pi \end{aligned} \quad (20)$$

## 3. Application to fiber composite tubes

### 3.1. Tube made of an anisotropic material with different orientations

Recently Rohde et al. [6] have proposed a novel composite tube design which exhibits the bend–twist coupling described in the previous section (see Fig. 3). In this design two different lay-ups or fiber orientations are used for each half of the circular tube. Assume the tube is made of a unidirectional fiber composite. The fiber-angle is the angle between the fiber direction (1-direction) and the  $x$ -axis. It is assumed  $+\alpha$  for the top half of the tube (Material 1) and  $-\alpha$  for the bottom half (Material 2) as shown in Figs. 4 and 5.

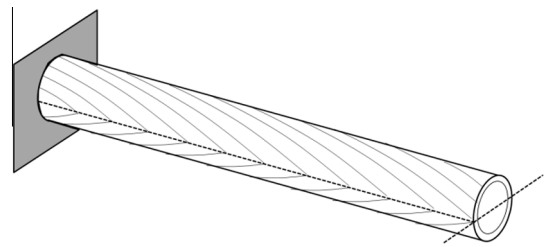


Fig. 3. An isometric view of the composite tube showing different fiber orientations on the top and bottom halves.

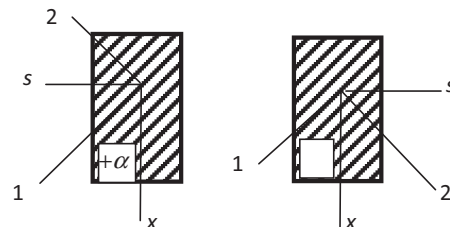


Fig. 4. The left figure shows the top half of the tube with fiber orientation  $+\alpha$ . Right figure shows the bottom half (fiber angle  $-\alpha$ ) viewed from the top. Note the difference in the direction of circumferential direction ( $s$ -axis) in the two figures.

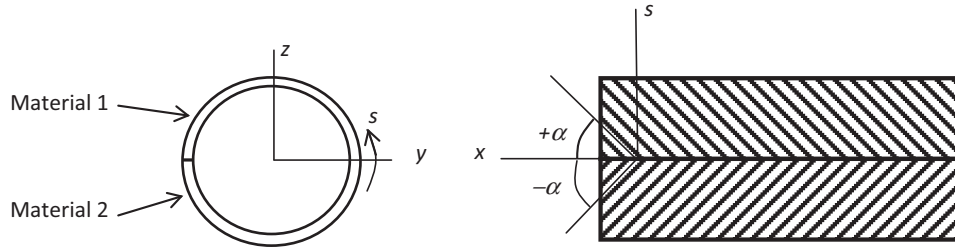


Fig. 5. Another view of the composite tube shown in Fig. 4 with opposing fiber angles in the top and bottom halves of the tube. Note that *s* is the circumferential direction.

The elastic constants in the *x*–*s* coordinate system can be obtained from the orthotropic engineering elastic constants  $E_1, E_2, G_{12}$  and  $\nu_{12}$  as follows [8]:

$$\begin{aligned} \frac{1}{E_x} &= \frac{1}{E_1} l^4 + \left( -\frac{2\nu_{12}}{E_1} + \frac{1}{G_{12}} \right) l^2 m^2 + \frac{1}{E_2} m^4 \\ \frac{1}{G_{xs}} &= 2 \left( \frac{2}{E_1} + \frac{2\nu_{12}}{E_1} + \frac{4\nu_{12}}{E_1} - \frac{1}{G_{12}} \right) l^2 m^2 + \frac{1}{G_{12}} (l^4 + m^4) \\ \frac{\eta_{x,xs}}{E_x} &= \left( \frac{2}{E_1} + \frac{2\nu_{12}}{E_1} - \frac{1}{G_{12}} \right) l^3 m + \left( -\frac{2\nu_{12}}{E_1} - \frac{2}{E_2} + \frac{1}{G_{12}} \right) l m^3 \\ l &= \cos \alpha, \quad m = \sin \alpha \end{aligned} \quad (21)$$

It is obvious from the above relations that for the present example

$$\begin{aligned} E_x^{(1)} &= E_x^{(2)} = \bar{E}_x, \quad \Delta E_x = 0 \\ G_{xs}^{(1)} &= G_{xs}^{(2)} = \bar{G}_{xs}, \quad \Delta G_{xs} = 0 \\ \eta_{x,xs}^{(1)} &= -\eta_{x,xs}^{(2)}, \quad \bar{\eta}_{x,xs} = 0, \quad \Delta \eta_{x,xs} = 2\eta_{x,xs}^{(1)} \\ \left( \frac{\eta_{x,xs}^2}{E_x} \right) &= \left( \frac{\eta_{x,xs}^{(1)}}{\bar{E}_x} \right)^2 \end{aligned} \quad (22)$$

Then Eqs. (8) and (13) can be simplified as

$$\begin{aligned} \bar{E}_x A \varepsilon_{x0} &= P \\ \bar{E}_x I \kappa_z &= M_y + 4R^2 h \eta_{x,xs}^{(1)} \tau_0 \\ \bar{E}_x I \kappa_y &= -M_z \\ \bar{\phi} &= \frac{2\eta_{x,xs}^{(1)}}{\pi} \kappa_z + \frac{1}{R} \left[ \frac{1}{\bar{G}_{xs}} - \frac{(\eta_{x,xs}^{(1)})^2}{\bar{E}_x} \right] \tau_0 \end{aligned} \quad (23)$$

From the above relations (2nd and 4th equations) one can note the coupling between the bending moment  $M_y$  and the torque  $T$ . The two relevant equations can be written as

$$\begin{aligned} \kappa_z &= \frac{M_y}{\bar{E}_x I} + \frac{2\eta_{x,xs}^{(1)}}{\pi \bar{E}_x I} T \\ \bar{\phi} &= \frac{4\eta_{x,xs}^{(1)}}{\pi \bar{G}_{xs} J} M_y + \left( 1 + \left( \frac{8}{\pi^2} - 1 \right) \eta_{x,xs}^{(1)} \eta_{xs,x}^{(1)} \right) \frac{T}{\bar{G}_{xs} J} \end{aligned} \quad (24)$$

In deriving the above relations we have used  $T = 2\pi R^2 h \tau_0$ ,  $J = 2I = 2\pi R^3 h$ , and the symmetry relation  $\eta_{x,xs}/E_x = \eta_{xs,x}/G_{xs}$ .

The shear stresses due to the transverse force  $F_z$  can be obtained from (18) using  $\Delta E_x = 0$  and  $E_x^{(1)} = E_x^{(2)} = E_x$ :

$$\tau_{xs}(\theta) = \tau_{xs}(0) + \frac{V_z R^2}{I} (\cos \theta - 1) \quad (25)$$

Note that the shear stress expression is the same for both top and bottom halves of the tube. The constant  $\tau_{xs}(0)$  can be evaluated from the fact that the moment of the shear stresses about the center should vanish as the force  $F_z$  is applied at the center. The final expression for shear stress takes the form

$$\tau_{xs}(\theta) = \frac{V_z R^2}{I} \cos \theta = \frac{V_z R}{I} y \quad (26)$$

Thus the shear stress distribution will be symmetric about the *z*-axis such that  $\tau_{xs}(y) = \tau_{xs}(-y)$ . Thus the shear flow will be in the counter clockwise direction on the right half of the tube ( $y > 0$ ) and in the clockwise direction in the left half ( $y < 0$ ). The shear stresses will not contribute to rotation about the *x*-axis as the material is also symmetric about the *z*-axis.

### 3.2. Shear center

Consider a cantilevered tube clamped at  $x = 0$ . First we will consider the case where the tube is subjected to a force  $F_z$  at the tip  $x = L$ . The force is such that the line of action is through the center of the tube. The bending moment distribution is given by  $M_y(x) = -F_z(L - x)$ . The tip rotation about the *x*-axis can be obtained from the second of Eq. (24):

$$\frac{d\psi_x}{dx} = \bar{\phi} = \frac{4\eta_{xs,x}^{(1)}}{\pi \bar{G}_{xs} J} F_z (x - L) \quad (27)$$

Integrating the above equation and noting  $\psi_x(0) = 0$  we obtain the tip rotation  $\psi_x^F$  due to the transverse force  $F_z$  as

$$\psi_x^F = \frac{-2\eta_{xs,x}^{(1)}}{\pi \bar{G}_{xs} J} F_z L^2 \quad (28)$$

From (24) the rotation  $\psi_x^T$  due to torque  $T$  can be derived as

$$\psi_x^T = \left( 1 + \left( \frac{8}{\pi^2} - 1 \right) \eta_{x,xs}^{(1)} \eta_{xs,x}^{(1)} \right) \frac{TL}{\bar{G}_{xs} J} \quad (29)$$

The location of the shear center can be derived as follows. Let the shear center distance – distance of the shear center from the tube axis – be denoted by  $e_y$  (Fig. 6). That is, if the transverse force  $F_z$  is applied at the shear center it would not produce any twisting of the tube, as the torque produced by the eccentric loading,  $F_z e_y$ , would cause an angle of twist equal in magnitude but opposite in direction to that produced by the force  $F_z$ . Then,

$$\psi_x^F = -\frac{\psi_x^T}{T} (F_z e_y) \Rightarrow e_y = -\frac{(\psi_x^F / F_z)}{(\psi_x^T / T)} \quad (30)$$

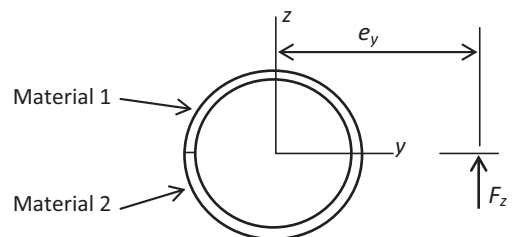


Fig. 6. Shear center for a tube made of an anisotropic material with different orientation of the principal material axis in the top and bottom halves of the tube as shown in Fig. 3.

Substituting from (28) and (29) in the above equation, the shear center distance can be written in a non-dimensional form as

$$\frac{e_y}{L} = \frac{-(\psi_x^F/F_z)}{(\psi_x^T/T)L} = \frac{2\eta_{xs,x}^{(1)}}{\pi\left(1 + \left(\frac{8}{\pi^2} - 1\right)\eta_{x,xs}^{(1)}\eta_{xs,x}^{(1)}\right)} \quad (31)$$

From (23) we note that the bending moment  $M_z$  due to a transverse force  $F_y$  will not cause any twisting. Hence the shear center will be on the  $y$ -axis at a distance  $e_y$  from the center of the tube.

We have already shown that the transverse shear stresses due to the force  $F_z$  do not contribute to the rotation due to symmetry of the across section about the  $z$ -axis. Thus we note the eccentricity of the shear center in the present case is due to the extension-shear coupling of the material. In homogeneous beams the eccentricity of the shear center is due to asymmetry of the cross section, e.g., a C-channel. That is why the shear center location in the present case is independent of the radius  $R$ , but proportional to the tube length  $L$  and the coefficients of mutual influence  $\eta$ .

### 3.3. Laminated composite tubes

In most applications the tube has to be made of multiple plies with different fiber orientations to achieve a given bending stiffness, torsional stiffness and desired bend–twist coupling. In that case the laminated tube can be modeled as a tube with an equivalent anisotropic material. The compliance matrix  $[\bar{S}]$  (see Eq. (4) above) of such an equivalent material can be obtained from the in-plane stiffness  $[A]$  of the laminate as follows [8]:

$$[\bar{S}_{eq}] = h[A]^{-1} \quad (32)$$

where  $h$  is the laminate thickness. It should be noted that the above idealization is valid only for thin-walled tubes such that  $h/R \ll 1$ .

### 3.4. Tapered tubes

In cantilevered tubes the bending moment varies linearly as  $M_y = F_z(x - L)$ . Hence, it makes sense to use a tapered tube for maximum efficiency. In the following we derive an expression for  $e_y$  for tapered tubes. Assume the radius of the tube varies as  $R(x) = R_0 + kx$ , where  $k = (R_L - R_0)/L$ . First we consider the case where only the transverse force  $F_z$  is acting and torque  $T = 0$ . From (24) the angle of rotation per unit length can be written as

$$\bar{\phi} = \frac{d\psi_x}{dx} = \frac{4\eta_{xs,x}^{(1)}}{\pi G_{xs} J(x)} F_z(x - L) \quad (33)$$

Note that  $J(x) = 2\pi R^3(x)h$ . Integrating the above equation and applying the BC  $\psi_x(0) = 0$  we obtain the angle of twist  $\psi_x^F$  at the tip of the tube due to the transverse force as

$$\psi_x^F = \frac{-\eta_{xs,x}^{(1)} F_z L^2}{2\pi^2 G_{xs} h R_0^2 R_L} \quad (34)$$

Similarly the unit angle of twist for a torque applied at the tip is given by (see Eq. (24))

$$\bar{\phi} = \left(1 + \left(\frac{8}{\pi^2} - 1\right)\eta_{x,xs}^{(1)}\eta_{xs,x}^{(1)}\right) \frac{T}{G_{xs} J(x)} \quad (35)$$

Note that  $J(x) = 2\pi R(x)^3 h$ . Integrating and applying the BC  $\psi_x(0) = 0$ , we obtain

$$\psi_x^T = \left(1 + \left(\frac{8}{\pi^2} - 1\right)\eta_{x,xs}^{(1)}\eta_{xs,x}^{(1)}\right) \frac{TL(R_0 + R_L)}{4\pi G_{xs} R_0^2 R_L^2 h} \quad (36)$$

where the superscript  $T$  denotes the angle of twist due to torque. Then the location of the shear center can be derived as before:

$$\frac{e_y}{L} = \frac{-(\psi_x^F/F_z)}{(\psi_x^T/T)L} = \frac{4\eta_{xs,x}^{(1)}}{\pi\left(1 + \left(\frac{8}{\pi^2} - 1\right)\eta_{x,xs}^{(1)}\eta_{xs,x}^{(1)}\right)\left(1 + \frac{R_0}{R_L}\right)} \quad (37)$$

Again we note that the shear center distance is independent of the radius of the tube but proportional to the tube length  $L$ . It also depends on  $\eta$ 's and the amount of taper defined by the ratio  $R_0/R_L$ .

## 4. Finite element analysis

The commercial finite element software Abaqus was used for computational analysis of aforementioned tubes. Eight-node doubly curved thick shell elements (S8R Element) were used to model the tubes. This element has six DOFs per node. About 30 elements were used along the circumference of the tube. The number of elements in the length direction was such that the aspect ratio of elements is approximately equal to unity. That is, the elements were almost square in shape. In the examples considered the tube was fixed at one end by setting all degrees of freedom equal to zero. At the free end, the center a reference node was created at the center of the tube and it was connected to the circumferential nodes using multipoint constraints. The transverse force and the couple were applied at the reference node. The finite element model including the deformed shape of the tube is shown in Fig. 7.

## 5. Results and discussion

All examples below, except Example 1C which is for a tapered tube, are concerned with a cantilevered tube of length  $L$ , mean

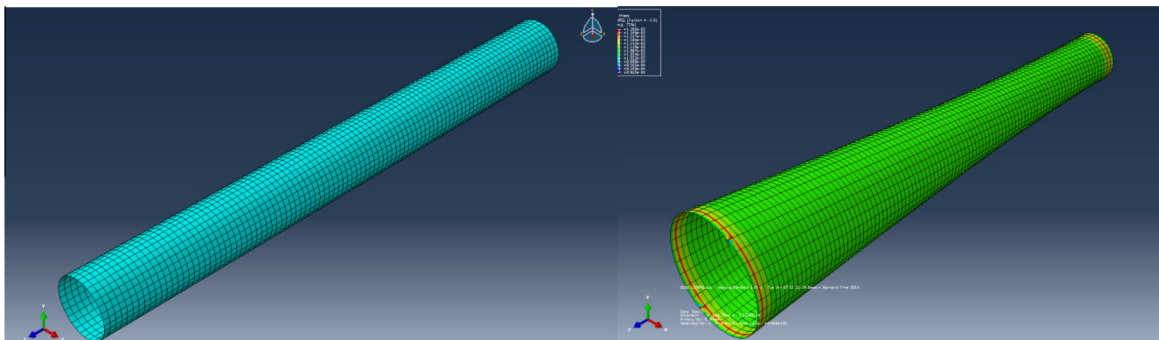


Fig. 7. (a) FE mesh of the composite tube; (b) deformed shape after subjected to a torque at the tip. Note that the circumferential expansion of the tube is exaggerated in the figure due to magnification of displacements.

**Table 1**

Results for tubes made of one orthotropic material but with opposite fiber orientations in the top and bottom halves. The top half of the tube consists of [+20] layers and the bottom half of [-20] layers.

Length $L$ (mm)	Load $F_z$ (N) $T$ (N-mm)	Tip deflection $w_0(L)$ ( $10^{-3}$ mm)		Tip rotation $\psi_x(L)$ ( $10^{-6}$ Radians)		Shear center $e_y/L$	
		FEA	Analytical	FEA	Analytical	FEA	Analytical
		200	$F_z = 1$ $T = 1$	9.162 77.14	9.211 78.71	77.21 1.915	78.70 1.951
300	$F_z = 1$ $T = 1$	30.71 174.0	31.11 177.0	174.0 2.881	177.0 2.920	0.202	0.202

**Table 2**

Results for tubes made of two different composite laminates. The lay-up for the top half of the tube is [0/20<sub>2</sub>/0]<sub>S</sub>; for the bottom half [0/-20<sub>2</sub>/0]<sub>S</sub>.

Length $L$ (mm)	Load $F_z$ (N) $T$ (N-mm)	Tip deflection $w_0(L)$ ( $10^{-6}$ mm)		Tip rotation $\psi_x(L)$ ( $10^{-6}$ Radians)		Shear center $e_y/L$	
		FEA	Analytical	FEA	Analytical	FEA	Analytical
		200	$F_z = 1$ $T = 1$	4.462 25.23	4.254 25.49	25.23 1.539	25.49 1.555
300	$F_z = 1$ $T = 1$	14.610 56.8	14,350 57.36	56.83 2.31	57.36 2.332	0.0820	0.0820

radius  $R = 10$  mm and wall thickness  $h = 2$  mm. The tapered tube's radius varies from 5 mm at the fixed end to 15 mm at the free end.

### 5.1. Example 1A – one anisotropic material

The laminate configuration or lay-up is denoted by [20\*]. The superscript \* denotes that for the top half of the tube  $\alpha = +20^\circ$  and for the bottom half  $\alpha = -20^\circ$  (see Fig. 3). The elastic constants are:  $E_1 = 138$  GPa;  $E_2 = 9$  GPa;  $G_{12} = 6.9$  GPa and  $\nu_{12} = 0.3$ . Tubes of two different lengths, 200 and 300 mm, were considered to demonstrate the length-dependence of shear center location. Two different forces were considered: a transverse tip force  $F_z$  applied at the center of the tube and a torque  $T$ . In Table 1 the FE results are compared with that obtained from the analytical method. It can be noted that the agreement for deflection, rotation and the shear center distance are excellent. Furthermore, the ratio  $e_y/L$  does not change at all for the two different lengths of the tube.

### 5.2. Example 1B – laminated tube

In this example we consider a laminated tube. The lay-up is denoted by [0/(20\*)<sub>2</sub>/0]<sub>T</sub>. Explicitly stated, the top half of the tube has a lay-up given by [0/(20)<sub>2</sub>/0]<sub>T</sub> and the bottom half [0/(-20)<sub>2</sub>/0]<sub>T</sub>. The elastic constants of the ply material were same as in Example 1A. The forces applied are similar to the previous example. The results are presented in Table 2. Again one notices that the agreement between the analytical results and FEA results is excellent. The shear center distance specified by  $e_y/L$  is smaller, i.e., the shear center is closer to the tube center, for the laminated tube compared to the 20-degree lamina in Example 1A, because

the laminate includes some 0-degree plies. The effective coefficient of mutual influence for the laminate is smaller than that for the 20-degree ply. A simple calculation shows that  $|\eta_{x,xx}| = 1.79$  and  $|\eta_{x,xy}| = 0.286$  for the 20-degree lamina, whereas for the above laminate  $|\eta_{x,xx}| = 1.26$  and  $|\eta_{x,xy}| = 0.125$ . This reduction in the effective coefficient of mutual influence due to the presence of 0-degree plies reduces the shear center distance also.

### 5.3. Example 1C – tapered laminated tube

The material properties and laminate configurations are same as in Example 2 but the laminate is tapered with  $R_0 = 5$  mm and  $R_L = 15$  mm. The results are presented in Table 3. The agreement between analytical and finite element analysis is excellent. One can note from Eq. (37) that the ratio  $e_y/L$  depends only on the effective coefficients of mutual influence and the ratio  $R_0/R_L$ . This is confirmed by the FEA results also.

## 6. Summary

An analytical model is presented for thin composite tube subjected to combined bending and torsion. The methods are applied to a special design of composite tubes with different lay-ups in the top and bottom halves of the tube cross section. Due to the difference in lay-ups, the tubes exhibit strong bend–twist coupling and the shear center is away from the center of the tube. The shear center distance is independent of the tube radius but is proportional to the length of the tube and the effective coefficient of mutual influence of the laminates. The results are verified by using finite

**Table 3**

Results for tapered laminated tubes.

Length $L$ (mm)	Load $F_z$ (N) $T$ (N-mm)	Tip deflection $w_0(L)$ ( $10^{-6}$ mm)		Tip rotation $\psi_x(L)$ ( $10^{-6}$ Radians)		Shear center $e_y/L$	
		FEA	Analytical	FEA	Analytical	FEA	Analytical
		200	$F_z = 1$ $T = 1$	2,774 22.38	2,676 22.66	22.38 2.716	22.66 2.764
300	$F_z = 1$ $T = 1$	9,148 50.37	9,034 50.99	50.37 4.080	50.99 4.146	0.0411	0.0410

element analysis of the tubes considered. The excellent agreement in results suggests that the assumptions made in the analytical model are reasonable and correct.

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